UNIT-I : INTRODUCTION TO PROBABILITY

Theory of probability :- probability is a statistical tool for measuring the chance or happening an event or an incident. It is introduced in the 17th century.

This concept has a wide number of applications in statistics, natural sciences, social sciences.

* **Meaning and definition of probability :-** The term probability refers to an event the happening and non-happening of which is uncertain. Literally, it means a chance, a possibility, a likelihood.

Example :

i) possibly it will rain today. iii) this year profit likely the preceding years.

ii) there is chance of your getting first class.

Mathematically, it is a number which is expressed either in the form of a fraction, a percentage or a decimal.

The value of the probability range from 0 to 1 and it is never negative.

Definitions :-

Experiment (Trial) : Any operation that results in two or more outcomes is called an experiment.

Conducting any experiment at one time is called as a trial.

Example : Tossing a fair coin is an experiment (expressed) and has two possible outcomes head or tail.

* **Random experiment :** Random experiment is an experiment in which the results (outcomes) are uncertain but may be any one of the possible outcomes.

Example :

1) Tossing of a coin is a Random experiment.

2) Any game is a Random experiment.

Outcome : the resultant (result) of a random experiment is known as outcome.

Example : In tossing a coin, head and tail are outcomes.

Simple events (Events) : the possible outcomes of a random experiment are called simple events.

Any set of simple events is called an event.

Example : In rolling a dice getting an even number {2,4,6} getting a odd number {1,3,5} getting a prime number {2,3,5}

Elementary Event (or) simple event : A simple event or elementary event is an event which cannot be broken or divided further into smaller events.

example : In tossing a coin head and tail are simple events.

Equally likely events : In a trial the events are said to be equally likely events if they have equal chance to occur.

Example : In tossing a coin getting a Head (H), getting a tail (T) are equally likely events.

Exhaustive (total) Events : the total number of possible outcomes in a random experiment is known as exhaustive events, and is denoted by 'N'.

These are always positive integers.

Example : In tossing a coin, total number of cases is N = 2

Sample space : events can be treated as sets. The set of all possible outcomes is called the sample space of the trial, denoted by 'S'. Clearly $n(S) = N$ = total number of cases

Example : In tossing a coin sample space $S = \{H, T\}$
$n(S) = 2$

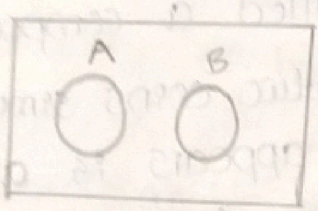
Favourable events (outcomes) : The cases which can produce a required event to us are called as favourable cases.

Example : In throwing of two dice, the number of cases, favourable for getting the sum 5 is
(1,4) (4,1) (2,3) (3,2) i.e 4

Mutually exclusive events :- No two or more events can occur in the same trial are called mutually exclusive events.

Two events are said to be MEE, if the happening of one event prevents the happening of the other events.

In otherwords, two MEE cannot occur simultaneously.



A, B are two MEE $A \cap B = \phi$

In this case $P(A \cup B) = (P(A)) + (P(B)) = P(A) + P(B)$ only

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Example : In tossing a coin getting head, tail are two MEE

Independent Events : Two or more events are said to be independent if the happening or non-happening of one event is not affected by the happening or non-happening of the other events.

Example: In tossing a coin twice, getting a head in the first trial is independent of getting a head in the second trial. If A and B are independent then $P(A \cap B) = P(A)\,P(B)$

Dependent events :- Two events are said to be dependent if the occurance of one event depends on the occurance of the other event.

Example: (If we draw a card from a pack of cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But, however) if the first card drawn is not replaced then the second draw is dependent on the first draw.

Mathematically if A and B are dependent events

$$P(A \cap B) \neq P(A)\,P(B)$$

$$P(A \cap B) = P(A)\,P(B/A)$$

with replacement = Independent events

without replacement = Dependent events

Compound Event: The joint occurance of two or more simple events is called a compound event.

Example: Tossing of two coins simultaneously, the event at least one head appears is a compound event.

$S = \{HH, HT, TH, TT\}$

AB, $A \cap B$, A and B becomes the popular notations for compound events.

Complementary events: The event A is called complementary event of $\bar{A}$. When a dice is thrown the occurance of A = Even number = $\{2, 4, 6\}$.

$\bar{A}$ = Odd number = $\{1, 3, 5\}$ are complementary events.

A', $\bar{A}$, A^c becomes the popular notations for complementary events

Probability space: The triplet (S, B, P) is called probability space

where S = sample space

B = Borel σ-field is set of all subsets of S (i.e contain all events)

P = probability function defined on Borel σ-field

Internal

Imp 10M (or) 5M

Various Types of definitions on Probability and its criticism:

Generally probability nothing but chance of happening of an event. There are three types of definitions for the happening of an event. they are

1) Mathematical or Apriori or classical or Bernouli definition of probability:

Suppose E is an event of a trial. If the total number of cases of the trial is N in which M are favourable to the event E such that the N events are equally likely and mutually exclusive. then the probability of happening the event E is defined as,

$$P[\text{an Event } E] = \frac{\text{Favourable number of cases}}{\text{Exhaustive (Total) number of cases}} = \frac{M}{N}$$

, It is called the success probability of E is denoted by P.

The probability of non-happening of E is called failure probability of E is denoted by $P(\bar{E})$ or q

$$P(\bar{E}) = \frac{N-M}{N} = 1 - \frac{M}{N} = 1 - P(E)$$

$$P(E) + P(\bar{E}) = 1 \quad (\text{or}) \quad p + q = 1$$

the numerical value of its probability always lies in between 0 and 1.

$$0 \le P(E) \le 1$$

Criticism :- merits & demerits

1) Mathematical definition on probability becomes easy to understand.
2) It becomes easy to calculate.
3) It is more real and practical.
4) It can be used for solving numerical problems.
5) this definition holds good if number of cases is finite.
6) If number of cases is infinite in such cases this definition does not holds good.
7) this definition holds good iff all cases are mutually exclusive, equally likely and exhaustive.

2) Statistical or posteriori or Empherical or von mison definition of probability :

This definition of probability is not based on logic but past experience and experiment and present conditions.

If an experiment is repeated large number of times (as $n \to \infty$) independently under certain identical conditions and if m cases are favourable for the happening of an event E then the limit of the ratio $\frac{m}{n}$, if it exists is called as probability of happening of an event E respectively.

symbolically, if event E happens m times out of total 'n' trials, then

$$P(E) = p = \lim_{n \to \infty} \frac{m}{n}$$

For example, if a coin is tossed 100 times and the head turn up 55 times, then the relative frequency of head will be $\frac{55}{100} = 0.55$

Criticism :

i) Statistical definition can be used in imaginary experiments. m

ii) It gives us more reliable results when we repeat an experiment more number of times. m

iii) Statistical definition requires identical conditions but it becomes impossible to have identical conditions for every repetition. In such cases this definition does not hold good. D

iv) It is a time taking. D

v) $\lim_{n \to \infty} \frac{m}{n}$ may not exists always. D

vi) It cannot be popularly used for solving practical problems. D

3) Rules
Axiomatic Definition of Probability or Bernstern definition of probability or Axioms of probability :-

Let (S, B, P) be a probability space. A function P defined on σ - field B satisfying the following axioms

i) $P(E_i) > 0 \quad \forall\, i$ (Positively)

ii) $P(S) = 1$ (certainty)

iii) If the events $E_1, E_2 \ldots E_n$ ($\in B$) belongs to are disjoint events, then

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n)$$

i.e $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$ (additivity) the probability p satisfying the axioms positivity certainty and additivity is called axiomatic definition of probability.

Criticism:

i) Axiomatic definition plays a very important role in set theory of mathematics.

ii) so many probability distributions were developed on the basis this approach.

iii) It fail to specify particular practical method for the required event.

iv) It becomes less popular than mathematical definition of probability.

Properties of probability:-

i) For any event A $(P(A)) \geq 0$.

ii) For the impossible event $P(\phi) = 0$

iii) the probability of the sample space, certain event is unity. i.e $P(S) = 1$

iv) For any event A, probability of the complementary event of A is $P(\bar{A}) = 1 - P(A)$

v) If A and B are mutually disjoint events then $P(A \cup B) = P(A) + P(B)$

vi) For any two events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional probability :- If A and B are any two events then happening of event B after the event A has already happened is called the conditional event of B given A and is denoted by B/A.

Happening of event A when the event B has already happened is known as conditional event of A given B and is denoted by A/B.

(the probability of a dependent event is called a conditional probability.

If there are two dependent events say A and B. the probability of happening the event A when the event B has already happened is called conditional probability of A given B. It is denoted by $p(A/B)$ and is given by $p(A/B) = \frac{p(A\cap B)}{p(B)}$, $p(B)>0$

$$p(A\cap B) = p(B)\, p(A/B)$$

the probability of happening the event B when the event A has already happened is called the conditional probability of B given A.)

It is denoted by $p(B/A)$ and is given by

$p(B/A) = \frac{p(A\cap B)}{p(A)}$ $(p(A)>0)$ $p(A\cap B) = p(A)\, p(B/A)$

If there are three dependent events say A, B and C $p(A\cap B\cap C) = p(A)\, p(B/A)\, p(C/A\cap B)$

Note:

1) $p(A/A) = 1$ $p(A/A) = \frac{p(A\cap A)}{p(A)} = \frac{p(A)}{p(A)} = 1$

2) A, B are MEE's then $p(A/B)=0$ $p(A/B) = \frac{p(A\cap B)}{p(B)} = \frac{0}{p(B)} = 0$

3) A, B are independent events

$p(A/B) = p(A)$ $\Rightarrow \frac{p(A\cap B)}{p(B)} = \frac{p(A)\cdot \cancel{p(B)}}{\cancel{p(B)}} = p(A)$

$p(B/A) = p(B)$ $\Rightarrow \frac{p(A\cap B)}{p(A)} = \frac{\cancel{p(A)}\cdot p(B)}{\cancel{p(A)}} = p(B)$

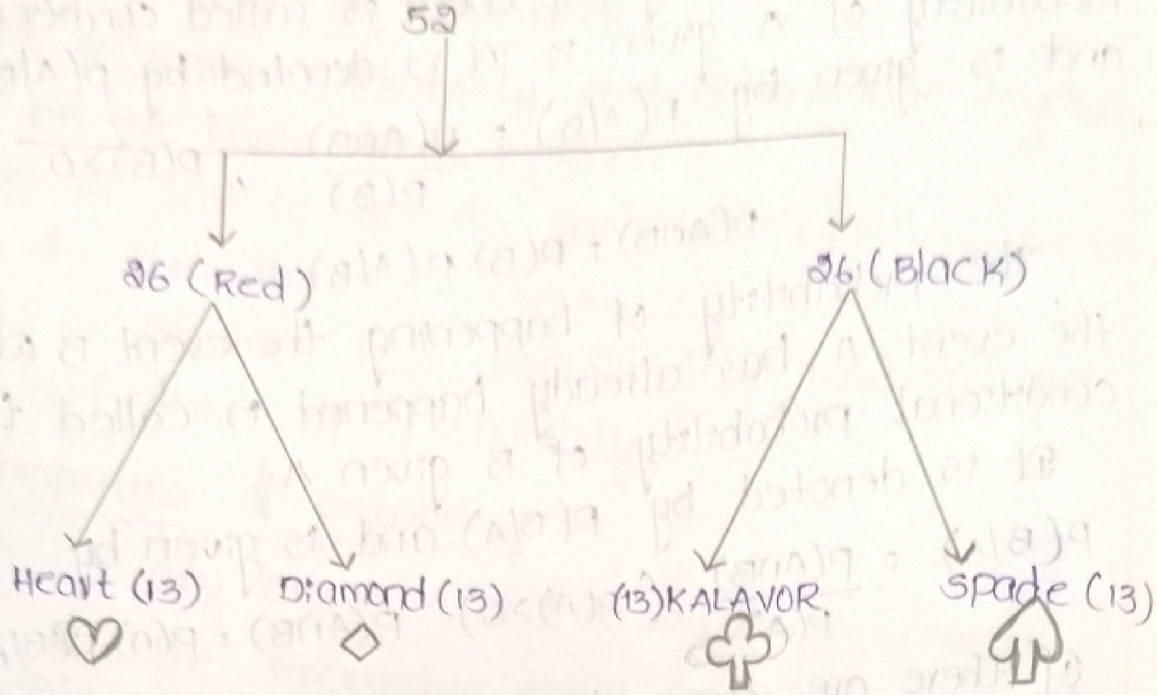
$p(A\cap B) = p(A)\, p(B)$

permutations and combinations:- the word permutation refers to arrangement. And the word combination refers to groups. these terms are used in the calculation of probability.

$^nP_r = \frac{n!}{(n-r)!}$ $\quad$ $^{10}P_3 = \frac{10!}{7!} = 720 = \frac{10\times9\times8\times\cancel{7!}}{\cancel{7!}}$

$^nC_r = \frac{n!}{r!(n-r)!}$ $\quad$ $^{10}C_3 = \frac{10!}{3!7!} = 120 = \frac{10\times9\times8\times\cancel{7!}}{3\times2\times1\times\cancel{7!}}$

Playing cards :-



In each suit we have 13 cards :-
2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q, A

J = Jackie = 4 $\quad$ K = King = 4 $\quad$ Q = Queen = 4

A = Ace = 4

Note:

* $0! = 1$
* $^nC_0 = 1$
* $^nC_n = 1$
* $^nC_0 = {^nC_n} = 1$

Calculation of probability of an event :-

probability of happening an event E =

$$E = \frac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}}$$

$$P = \frac{m}{n}$$

where m = Number of favourable cases

n = Number of exhaustive cases (Total no. of events)

The probability of an event E will not happen = q

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

The probability can never be less than '0' nor it can be greater than '1'

Hence $0 \leq p \leq 1 \quad 0 \leq q \leq 1$

Problems on coin :-

1) Find the probability of getting a head in tossing a coin?

Sol:- If a coin is tossed sample space $S = \{H, T\}$

$n(S) = 2 = n$

Let E be the event getting a head

$E = \{H\}$ $n(E) = 1 = m$

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{Number of favourable events}}{\text{Number of exhaustive events}}$$

$$= \frac{m}{n} = \frac{1}{2}$$

2) If two coins are tossed simultaneously. Find the chance of getting atleast one tail.

Sol:- If two coins are tossed sample space = S

$S = \{HH, HT, TH, TT\}$

$n(S) = 4 = n$

Let E getting atleast one tail

$E = \{HT, TH, TT\}$ $n(E) = 3 = m$

Required probability $= p = P(E) = \frac{m}{n} = \frac{3}{4}$

Second method : probability of atleast one tail

$= 1 - P(\text{all heads})$

$= 1 - \frac{1}{4}$

$= \frac{3}{4}$

3) Find the chance of getting atleast one head five coins are tossed.

Sol: The probability of atleast one head = 1-P(all tails)

$= 1 - \frac{1}{2^5} = 1 - \frac{1}{32} = \frac{31}{32}$

Note: When a coin is tossed n times or n coins tossed simultaneously sample space $n(S) = 2^n$

4) Find the probability of getting one head in tossing two coins?

$S = \{HH, HT, TH, TT\}$ $n(S) = 4 = n$

Sol: $P(E) = \frac{2}{4} = \frac{1}{2}$ Let E getting one head $E = \{HT, TH\}$ $n(E) = 2 = m$

$P(E) = \frac{m}{n} = \frac{2}{4} = \frac{1}{2}$

Imp 5) Three unfaced coins are tossed, what is the probability of obtaining a) all tails b) one head c) atleast one head

Sol: Three coins are tossed sample space S

$S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$

$n(S) = 8$

a) Let A be the event getting all tails

$A = \{TTT\}$

$n(A) = 1$

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$

b) Let B be the event getting one head

$B = \{TTH, THT, HTT\}$

$n(B) = 3$

$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{3}{8}$

c) Let C be the event getting atleast one head

$C = \{TTH, THT, HTT, HHT, HTH, THH, HHH\}$

$n(C) = 7$

$P(C) = \frac{n(C)}{n(S)} = \frac{7}{8}$

6) If three coins are tossed find the probability of getting

i) Three heads ii) Two heads iii) no head

Sol. i) $P(A) = \frac{1}{8}$

ii) $P(B) = \frac{3}{8}$

iii) $P(C) = \frac{1}{8}$

Problems on dies:-

1) Find the probability of getting 2 in throwing a die?

Sol. The die has six sides

Sample space = S = {1, 2, 3, 4, 5, 6}

$n(S) = 6 = n$

Let E be the event getting the number 2

E = {2}

$n(E) = 1 = m$

$P(E) = \frac{m}{n} = \frac{1}{6}$

2) Find the probability of throwing a dice

i) 4 ii) an odd number iii) an even number

iv) prime number v) Greater than 4

Sol. The die has six sides

S = {1, 2, 3, 4, 5, 6}

$n(S) = 6 = n$

i) Let A be the event getting the number 4

A = {4}

$n(A) = 1$

$P(A) = \frac{n(A)}{n(S)} = \frac{m}{n} = \frac{1}{6}$

ii) Let B be the event getting odd number

B = {1, 3, 5} $n(B) = 3 = m$

$P(B) = \frac{m}{n} = \frac{3}{6} = \frac{1}{2}$

iii) Let C be the event getting even number

$C = \{2, 4, 6\}$ $n(C) = 3$

$P(C) = \frac{n(C)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

v) Let E be the event getting greater than 4

$E = \{5, 6\}$ $n(E) = 2$

$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$

iv) Let D be the event getting prime number

$D = \{2, 3, 5\}$ $n(D) = 3 = m$, $P(D) = \frac{m}{n} = \frac{3}{6} = \frac{1}{2}$

3) Two dies are rolled find the probability that the sum of dots (number) on the faces that turn up is
i) 8 ii) 11 iii) equal numbers

Sol: Let S be the sample space when two dies are thrown

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

$n(S) = 36$

i) Let A be the event of getting a total score of Eight in an experiment.

$A = \{(2,6)\ (3,5)\ (4,4)\ (5,3)\ (6,2)\}$

$n(A) = 5$

$P(A) = \frac{n(A)}{n(S)} = \frac{5}{36} = 0.1388$

ii) Let B be the event of getting a total score of 11 in an experiment

$B = \{(5,6)\ (6,5)\}$

$n(B) = 2$

$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{2}{36} = 0.0555$

iii) Let C be the event of getting equal numbers.

$C = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$n(C) = 6$

$P(C) = \frac{n(C)}{n(S)} = \frac{6}{36} = \frac{1}{6} = 0.1666$

4) In a single throw of 3 dies. Find the probability of getting the same number of each of them

Sol. total number of cases $= 6 \times 6 \times 6 = 216$ $(n(S))$

the number of favourable cases $= 6$ $(n(E))$

$E = \{(1,1,1)\ (2,2,2)\ (3,3,3)\ (4,4,4)\ (5,5,5)\ (6,6,6)\}$

the required probability $= \frac{6}{216} = \frac{1}{36}$

5) Find the probability of throwing sum 7 with two dice.

Sol. $\frac{6}{36} = \frac{1}{6}$ (sum 7 if two dies are thrown)

$= 0.1666$

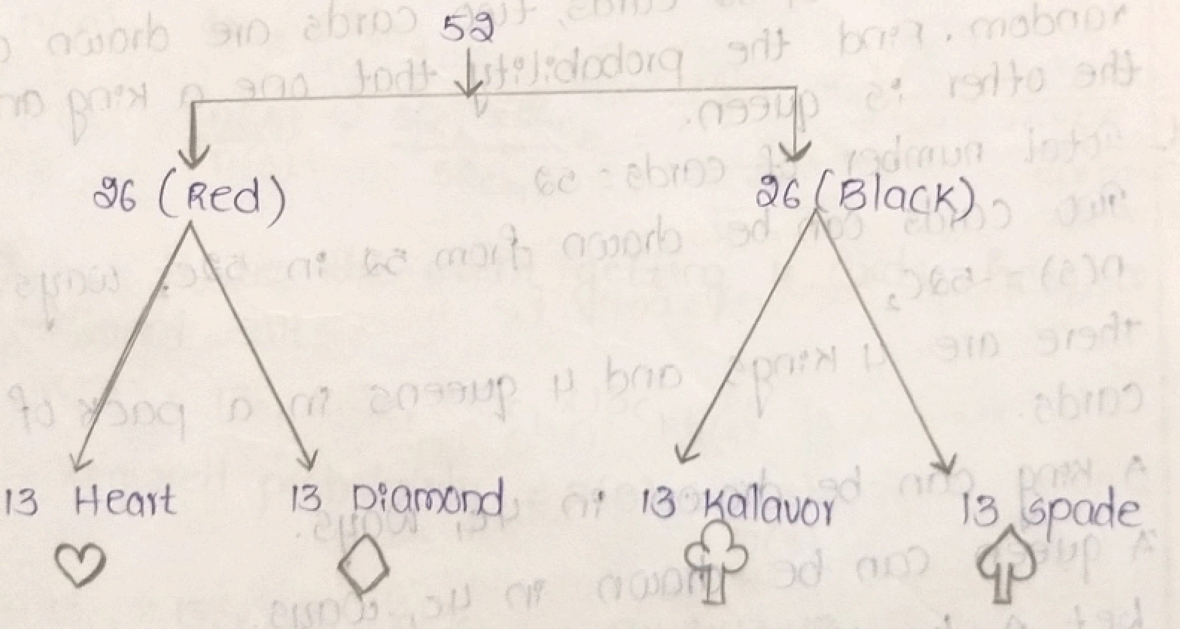
Use of combination in the theory of probability

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

${}^{n}C_{0} = 1 = {}^{n}C_{n} = 1 \qquad 0! = 1$

Cards



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2, 3, 4, 5 ..... 10, J, K, Q, A

J = Jackie = 4 K = King = 4 Q = Queen = 4 A = Ace = 4